

Metrics on the Real Quantum Plane

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Abstract

Using the frame formalism we determine some possible metrics and metric-compatible connections on the noncommutative differential geometry of the real quantum plane. By definition a metric maps the tensor product of two 1-forms into a 'function' on the quantum plane. It is symmetric in a modified sense, namely in the definition of symmetry one has to replace the permutator map with a deformed map σ fulfilling some suitable conditions. Correspondingly, also the definition of the hermitean conjugate of the tensor product of two 1-forms is modified (but reduces to the standard one if σ coincides with the permutator). The metric is real with respect to such modified \ast -structure.

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1 Introduction and notation

It is an old idea ^{1, 2} that a noncommutative modification of the algebraic structure of space-time could provide a regularization of the divergences of quantum field theory, because the representations of noncommutative ‘spaces’ have a lattice-like structure. The main aim of noncommutative geometry ³ is to endow such an algebra with additional structures (starting from a differential calculus), so as to build a bridge between the algebra and its ‘geometrical’ interpretation. Since there is no unique prescription how to do this, it is useful to test possible prescriptions first on simpler models.

In this paper we choose as a noncommutative space(time) algebra model the so-called real quantum or Manin plane ⁴, and as a differential calculus upon it the so-called Wess-Zumino calculus ^{5, 6}. The latter is characterized by the property that the relations defining the module of 1-forms are covariant under the action of the quantum group $SL_q(2)$ and homogeneous in the generators. We adopt the noncommutative geometry formalism of ^{7, 8, 9, 10, 11}.

We start with a brief description of the latter. Let \mathcal{A} be an algebra with differential calculus $\{\Omega^*(\mathcal{A}), d\}$ ³ (here $\Omega^*(\mathcal{A})$ denotes the algebra of differential forms on \mathcal{A} and d the exterior derivative acting on the latter) and suppose that the calculus has a *frame* ^{7, 10}, i.e. a basis of 1-forms θ^i ($i = 1, 2, \dots, n$) which commute with the elements of the algebra,

$$\theta^i f = f \theta^i. \quad (1.1)$$

The relation

$$df = \theta^i e_i f \quad (1.2)$$

(with $f \in \mathcal{A}$) defines a set of derivations e_i dual to θ^i , from which it follows that the module structure of $\Omega^1(\mathcal{A})$ is given by

$$f dg = \theta^i f e_i g, \quad dg f = \theta^i (e_i g) f.$$

We see that the \mathcal{A} -bimodule $\Omega^1(\mathcal{A})$ is free of rank n as a left or right module. It can therefore be identified with the direct sum

$$\Omega^1(\mathcal{A}) = \bigoplus_1^n \mathcal{A} \quad (1.3)$$

of n copies of \mathcal{A} . In this representation θ^i is given by the element of the direct sum with the unit in the i -th position and zero elsewhere. We shall refer to the integer n as the dimension of the geometry.

The wedge product π in $\Omega^*(\mathcal{A})$ fulfills relations of the form

$$\theta^i \theta^j \equiv \pi(\theta^i \otimes_{\mathcal{A}} \theta^j) = P^{ij}_{kl} \theta^k \theta^l \quad (1.4)$$

(we omit the symbol \wedge of the wedge product), where P is a projector

$$P^{ij}_{mn} P^{mn}_{kl} = P^{ij}_{kl} \quad (1.5)$$

with entries $P^{ij}_{kl} \in \mathcal{Z}(\mathcal{A})$. If in particular the wedge product is such that the θ^i anti-commute then P is the antisymmetric projector

$$P^{ij}_{kl} = \frac{1}{2}(\delta_k^i \delta_l^j - \delta_k^j \delta_l^i).$$

From (1.3) it follows immediately that the algebra and its differential calculus are related in a simple manner. Let Λ_P^* be the exterior algebra over \mathbb{C}^n with the wedge product defined by (1.4). Then with the identification (1.3) it follows that one can write

$$\Omega^*(\mathcal{A}) = \mathcal{A} \otimes \Lambda_P^*. \quad (1.6)$$

Since the exterior derivative of θ^i is a 2-form it can necessarily be written as

$$d\theta^i = -\frac{1}{2}C^i_{jk}\theta^j\theta^k.$$

where, because of (1.4), the structure elements can be chosen to satisfy the constraints

$$C^i_{jk}P^{jk}_{lm} = C^i_{lm}.$$

It will also be convenient to introduce the quantities

$$C^{ij}_{kl} = \delta^i_k \delta^j_l - 2P^{ij}_{kl}. \quad (1.7)$$

Then from (1.5) we find that

$$C^{ij}_{kl}C^{kl}_{mn} = \delta^i_m \delta^j_n. \quad (1.8)$$

For simplicity, we shall further assume that the e_i are inner derivations: $e_i f = [\lambda_i, f]$, $\lambda_i \in \mathcal{A}$. From the θ^i we can construct a 1-form

$$\theta = -\lambda_i \theta^i \quad (1.9)$$

in $\Omega^1(\mathcal{A})$ which plays the role of a Dirac operator ³:

$$df = -[\theta, f]. \quad (1.10)$$

One can show that from the general consistency of the differential calculus it follows that

$$2P^{ij}_{kl}\lambda_i\lambda_j - F^i_{kl}\lambda_i - K_{kl} = 0 \quad (1.11)$$

for some array of elements $F^i_{jk}, K_{kl} \in \mathcal{Z}(\mathcal{A})$. In the cases which interest us here the latter vanish.

In order to consistently define a covariant derivative we need to introduce ⁸ a flip σ , i.e. a \mathcal{A} -bilinear map

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{\sigma} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}). \quad (1.12)$$

In the case of the De-Rham calculus on the commutative algebra of functions on an ordinary manifold it reduces to $\sigma(\omega \otimes_{\mathcal{A}} \omega') = \omega' \otimes_{\mathcal{A}} \omega$. In terms of the frame it is given by $S^{ij}_{kl} \in \mathcal{Z}(\mathcal{A})$ defined by

$$\sigma(\theta^i \otimes_{\mathcal{A}} \theta^j) = S^{ij}_{kl} \theta^k \otimes_{\mathcal{A}} \theta^l.$$

A covariant derivative on the module $\Omega^1(\mathcal{A})$ is a map

$$\Omega^1(\mathcal{A}) \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}). \quad (1.13)$$

satisfying both a left and a right Leibniz rule. We use the ordinary left Leibniz rule and define the right Leibniz rule as

$$D(\xi f) = \sigma(\xi \otimes_{\mathcal{A}} df) + (D\xi)f \quad (1.14)$$

for arbitrary $f \in \mathcal{A}$ and $\xi \in \Omega^1(\mathcal{A})$. The connection 1-form $\omega^i_k \equiv \omega^i_{jk}\theta^j$ is defined by

$$D\theta^i = \omega^i_k \otimes_{\mathcal{A}} \theta^k. \quad (1.15)$$

We shall impose the condition

$$\pi \circ (\sigma + 1) = 0 \quad (1.16)$$

that the antisymmetric part of a symmetric tensor vanish. This can be considered as a condition on the product or on the flip. In ordinary geometry it is the definition of π ; a 2-form can be considered as an antisymmetric tensor. Because of this condition the torsion is a bilinear map¹¹. The most general solution can be written in the form

$$1 + \sigma = (1 - \pi) \circ \tau \quad (1.17)$$

where τ is an arbitrary \mathcal{A} -bilinear map. Suppose that τ is invertible. Then because of the identity

$$1 = \pi + (1 + \sigma) \circ \tau^{-1}$$

one can identify the second term on the right-hand side as the projection onto the symmetric part of the tensor product. The choice $\tau = 2$ yields the value $\sigma = 1 - 2\pi$. If τ is not invertible then there arises the possibility that part of the tensor product is neither symmetric nor antisymmetric. Condition (1.16) applied to the tensor product $\theta^i \otimes_{\mathcal{A}} \theta^j$ becomes

$$P^{ij}_{lm} + S^{ij}_{hk} P^{hk}_{lm} = 0. \quad (1.18)$$

If the flip is such that in (1.11) $F^i_{jk} = K_{kl} = 0$ one possible linear connection is

$$\omega^i_{jk} = \lambda_l (S^{il}_{jk} - \delta^l_j \delta^i_k). \quad (1.19)$$

The corresponding connection 1-form is given by

$$\omega^i_k = \lambda_l S^{il}_{jk} \theta^j + \delta^i_k \theta. \quad (1.20)$$

The curvature of the covariant derivative D defined in (1.19) can be readily calculated. One finds the expression

$$\frac{1}{2} R^i_{jkl} = S^{im}_{rn} S^{np}_{sj} P^{rs}_{kl} \lambda_m \lambda_p. \quad (1.21)$$

This can also be written in the form

$$\frac{1}{2} R^i_{jkl} = -S^{im}_{rn} S^{np}_{sj} S^{rs}_{uv} P^{uv}_{kl} \lambda_m \lambda_p.$$

In complete analogy with the commutative case a metric g can be defined as an \mathcal{A} -bilinear, nondegenerate map¹¹

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{g} \mathcal{A} \quad (1.22)$$

and as such it can ¹⁵ be used to define a ‘distance’ between ‘points’. It is important to notice here that the bilinearity is an alternative way of expressing locality. In ordinary differential geometry if ξ and η are 1-forms then the value of $g(\xi \otimes \eta)$ at a given point depends only on the values of ξ and η at that point. Bilinearity,

$$g(f\xi \otimes_{\mathcal{A}} \eta h) = f g(\xi \otimes_{\mathcal{A}} \eta) h \quad \forall f, h \in \mathcal{A},$$

is an exact expression of this fact. In general the algebra introduces a certain amount of non-locality via its nontrivial commutation relations and it is important to assure that all geometric quantities be just that nonlocal and not more. Without the bilinearity condition it is not possible to distinguish for example in ordinary space-time a metric which assigns a function to a vector field in such a way that the value at a given point depends only on the vector at that point from one which is some sort of convolution over the entire manifold.

We define frame components of the metric by

$$g^{ij} = g(\theta^i \otimes_{\mathcal{A}} \theta^j).$$

They lie necessarily in the center $\mathcal{Z}(\mathcal{A})$ of the algebra. The condition that (1.19) be metric-compatible can be written as ¹⁰

$$S^{im}_{ln} g^{np} S^{jk}_{mp} = g^{ij} \delta_l^k. \quad (1.23)$$

As a way to remember this seemingly odd condition introduce a ‘covariant derivative’ $D_i X^j$ of a ‘vector’ X^j . The covariant derivative $D_i(X^j Y)$ of the product of X^j by a ‘field’ Y must be then defined as

$$D_i(X^j Y) = D_i X^j Y + S^{jl}_{im} X^m D_l Y$$

since there is a ‘flip’ as the index on the derivation crosses the index on the first ‘vector’. If we apply again this rule to $Y = Y^k Z$, with Y^k also a ‘vector’ and Z another ‘field’ we find

$$D_i(X^j Y^k Z) = D_i(X^j Y^k) Z + S^{jl}_{im} X^m Y^p S^{kn}_{lp} D_n Z.$$

Since g^{jk} is a ‘tensor’, the ‘crossing rule’ is the same as for $X^j Y^k$:

$$D_i(g^{jk} Z) = (D_i g^{jk}) Z + S^{jl}_{im} g^{mp} S^{kn}_{lp} D_n Z.$$

Therefore (1.23) is equivalent to the usual condition,

$$D_i g^{jk} = 0,$$

that the connection be compatible with the metric.

We shall require that the metric be symmetric in the sense

$$g \circ \pi = 0 \quad (1.24)$$

that it annihilates the 2-forms. This condition applied to the tensor product $\theta^i \otimes_{\mathcal{A}} \theta^j$ becomes

$$P^{ij}_{lm} g^{lm} = 0. \quad (1.25)$$

Let us now briefly summarize the additional conditions which arise from the requirement of existence of $*$ -structures. Assume \mathcal{A} is a $*$ -algebra. If ^{12, 13} the $*$ -structure of $\mathcal{A} \equiv \Omega^0(\mathcal{A})$ can be extended to a $*$ -structure of $\Omega^*(\mathcal{A})$ and

$$(df)^* = df^* \quad (1.26)$$

the differential calculus is said to be real. A sufficient condition for (1.26) to hold ¹⁴ is that the λ_i are anti-hermitian (w.r.t. the $*$ of \mathcal{A}) and the θ^i are hermitean (w.r.t. the extension of $*$ to $\Omega^*(\mathcal{A})$), so that the ‘Dirac operator’ θ is anti-hermitean.

To obtain a real covariant derivative it is necessary first of all that the flip σ satisfies a reality constraint (see Ref. ¹⁴), which takes the simple form

$$(S^{ji}_{kl})^* S^{lk}_{mn} = \delta_m^i \delta_n^j \quad (1.27)$$

if $(\theta^i)^* = \theta^i$. Moreover, the connection 1-form ω_k^i and the flip σ must satisfy a condition ¹⁴ involving both, which we do not report here because it is automatically satisfied in the case of the connection (1.19). In order to define a real metric one has to use σ to impose the reality condition of Ref. ¹⁴, which takes the simple form

$$S^{ij}_{kl} g^{kl} = (g^{ji})^* \quad (1.28)$$

in the case of a real frame. This is a combination of a ‘twisted’ symmetry condition and the ordinary condition of reality on a complex matrix. It can also be written as an ordinary condition of symmetry and a ‘twisted’ definition of reality. The map σ is also involved ¹⁴ in the reality condition for the curvature or for the covariant derivative acting on tensor powers of $\Omega^1(\mathcal{A})$. The latter implies the former, and takes the form of the braid equation,

$$S_{12} S_{23} S_{12} = S_{23} S_{12} S_{23} \quad (1.29)$$

where

$$(S_{12})^{abc}_{def} := S^{ab}_{de} \delta_f^c, \quad (S_{23})^{abc}_{def} := \delta_d^a S^{bc}_{ef}$$

The ‘infinitesimal distance’ ds corresponding to the metric g is introduced through the relation

$$ds^2 = g_{ij} \theta^i \otimes_{\mathcal{A}} \theta^j, \quad (1.30)$$

where $g_{ij} \in \mathcal{A}$ are the matrix elements of the inverse matrix of $\|g^{ij}\|$. Every representation of \mathcal{A} yields a distance between ‘points’ because of (1.6). Let $dt = \xi_i \theta^i \in \Omega^1(\mathcal{A})$ be an exact form, which we can think of as an infinitesimal displacement along an axis t and suppose that $|p\rangle$ is a common eigenvector of all the ξ_i : $\xi_i |p\rangle = \tilde{\xi}_i |p\rangle$. This would be the case for example if only one of them is not equal to zero. We define the element of distance δs along the ‘coordinate’ t at the state $|p\rangle$ by the equation

$$(\delta s)^2 = \langle p | ds^2 | p \rangle = g^{ij} \tilde{\xi}_i \tilde{\xi}_j.$$

Let \hbar be the length scale at which points become fuzzy and K^{-1} the scale at which the curvature effects become important. The definition of g which we have given is unambiguous

but the interpretation of the norm $|\delta s|^2$ of an infinitesimal displacement as a distance can be only made within the range

$$\hbar << |\delta s|^2 << K^{-1}.$$

If the displacement is too small then the points are not defined; if it is too large then an integral must be taken. The second problem was solved by Leibniz/Newton; the first is a feature, not a bug, of noncommutative geometry. We are especially interested in the region $|\delta s|^2 \simeq \hbar$ where the noncommutative effects become of interest.

There exist other definitions of distance. One proposal^{16, 17, 18} uses the Dirac operator to define distance on the space of pure states. Several authors^{19, 20} do not consider the bilinearity condition we have imposed as important and several^{21, 22, 23, 24, 25, 26, 27} consider the invariance under the coaction of a quantum group as essential.

It is sometimes convenient to write the metric as a sum

$$g^{ij} = g_S^{ij} + g_A^{ij}$$

of a symmetric and an antisymmetric part (in the usual sense of the word) The inverse matrix we write as a sum

$$g_{ij} = \eta_{ij} + B_{ij}$$

of a symmetric and an antisymmetric term. We shall choose as normalization when possible the condition that η_{ij} be the standard Minkowski or euclidean form.

2 The Wess-Zumino calculus

The extended real quantum plane is the $*$ -algebra \mathcal{A} generated by hermitian elements $(x^i) = (x, y)$,

$$x^* = x \quad y^* = y, \quad (2.1)$$

together with their inverses, fulfilling the relation

$$xy = \tilde{q}yx \quad (2.2)$$

with $|\tilde{q}| = 1$ and $q \neq \pm 1$, as well as the usual relations between inverses. We call it *extended* because in the original version⁴ the inverses x^{-1}, y^{-1} were not included; the word *real* refers to the $*$ -structure (2.1). The center of \mathcal{A} is trivial, $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$. We now show how the Wess-Zumino calculus⁵ fits in the scheme described in the previous section. We define, for $\tilde{q}^4 \neq 1$,

$$\lambda_1 = -\epsilon_1 \frac{\tilde{q}^4}{\tilde{q}^4 - 1} x^{-2} y^2, \quad \lambda_2 = \epsilon_2 \frac{\tilde{q}^2}{\tilde{q}^4 - 1} x^{-2}.$$

There is an ambiguity in this definition due to the fact that the defining relations (2.2) are homogeneous and which we reduce to a sign: $\epsilon_a = \pm 1$. The extra minus is a ‘historical convenience’. The important fact is that the λ_a are singular in the limit $\tilde{q} \rightarrow 1$ and that they are anti-hermitian if \tilde{q} is of unit modulus, as we are assuming. We find for $\tilde{q}^2 \neq -1$

$$\begin{aligned} e_1 x &= \epsilon_1 \frac{\tilde{q}^2}{(\tilde{q}^2 + 1)} x^{-1} y^2, & e_1 y &= \epsilon_1 \frac{\tilde{q}^4}{\tilde{q}^2 + 1} x^{-2} y^3, \\ e_2 x &= 0, & e_2 y &= -\epsilon_2 \frac{\tilde{q}^2}{\tilde{q}^2 + 1} x^{-2} y. \end{aligned} \quad (2.3)$$

These derivations are again extended to arbitrary polynomials in the generators by the Leibniz rule. Using them and (1.9), (1.10) we find

$$dx = \frac{\tilde{q}^2}{(\tilde{q}^2 + 1)} x^{-1} y^2 \epsilon_1 \theta^1, \quad dy = \frac{\tilde{q}^2}{\tilde{q}^2 + 1} x^{-2} y (\tilde{q}^2 y^2 \epsilon_1 \theta^1 - \epsilon_2 \theta^2) \quad (2.4)$$

and solving for the θ^i we obtain

$$\epsilon_1 \theta^1 = (\tilde{q}^2 + 1) xy^{-2} dx, \quad \epsilon_2 \theta^2 = -(\tilde{q}^2 + 1) x(xy^{-1} dy - dx).$$

The module structure which follows from the condition (1.1) that the θ^i commute with the elements of the algebra is equivalent to the Wess-Zumino relations ⁵

$$\begin{aligned} xdx &= \tilde{q}^2 dx, & xdy &= \tilde{q} dy + (\tilde{q}^2 - 1) dxy, \\ ydx &= \tilde{q} dxy, & ydy &= \tilde{q}^2 dy. \end{aligned} \quad (2.5)$$

One can show that they are invariant under the coaction of the quantum group $SL_q(2, \mathbb{C})$. This invariance was encoded in the choice of λ_a .

Consider the elements

$$u := \epsilon_2 \tilde{q}^{-2} x^2, \quad v := \epsilon_1 x^2 y^{-2}. \quad (2.6)$$

We shall see that each of the four possible choices of sign pairs corresponds to an identification of x and y as the coordinates of one of the four regions on \mathbb{R}^2 defined by the light cone of a metric with Minkowski signature. The u, v fulfill the quadratic commutation relation

$$uv = qvu \quad (2.7)$$

where $q := \tilde{q}^{-4}$. They and their inverses generate a slightly smaller algebra than \mathcal{A} . One also finds that (2.5) becomes

$$\begin{aligned} udu &= q^{-1} duu, & udv &= qdvu, \\ vdu &= q^{-1} dvv, & vdv &= qdvv. \end{aligned} \quad (2.8)$$

In terms of the new generators the θ^i become

$$\theta^1 = q^{-1} vu^{-1} du, \quad \theta^2 = uv^{-1} dv. \quad (2.9)$$

What we have done in fact is use the λ_a^{-1} as generators of the algebra and the differential calculus; otherwise nothing has been changed. The form θ is most conveniently expressed in terms of the λ_a . Since

$$\lambda_1 = \frac{1}{1 - q^{-1}} v^{-1}, \quad \lambda_2 = -\frac{1}{1 - q^{-1}} u^{-1} \quad (2.10)$$

we find that

$$\theta = \frac{1}{1 - q} (u^{-1} du - qv^{-1} dv).$$

It is an anti-hermitian closed form with vanishing square,

$$d\theta = 0, \quad (\theta)^2 = 0. \quad (2.11)$$

The volume element is a product of two exact forms:

$$\theta^1 \theta^2 = dudv.$$

The structure of the exterior algebra is given by the relations

$$(\theta^1)^2 = 0, \quad (\theta^2)^2 = 0, \quad \theta^1 \theta^2 + q \theta^2 \theta^1 = 0. \quad (2.12)$$

This can be written in the form (1.4) with

$$P = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -q & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (2.13)$$

If we reorder the indices $(11, 12, 21, 22) = (1, 2, 3, 4)$ then the C^{ij}_{kl} introduced in (1.7) is given by the expression

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

That is, $C^{12}_{21} = q$ and $C^{21}_{12} = q^{-1}$.

The reality of the differential implies that the structure elements must satisfy the conditions

$$((C^i_{jk})^* + C^i_{jk}) P^{jk}_{lm} = 0$$

from which it follows that

$$(C^i_{21})^* = -C^i_{12} = q^{-1} C^i_{21}, \quad (C^i_{12})^* = -C^i_{21} = q C^i_{12}.$$

More precisely, the independent coefficients are given by

$$C^1_{12} = (q^{-1} - 1) \lambda_2, \quad C^2_{12} = (q^{-1} - 1) \lambda_1. \quad (2.14)$$

The C^i_{jk} do not depend on the sign ambiguities. With the generators

$$t = \frac{1}{\sqrt{2}}(u + v), \quad r = \frac{1}{\sqrt{2}}(u - v) \quad (2.15)$$

the four possible sign combinations can be written as

$$\epsilon_1 = \epsilon_2 : \quad \text{sgn}(t) = \epsilon_1, \quad \epsilon_1 = -\epsilon_2 : \quad \text{sgn}(r) = \epsilon_2.$$

We shall later in Section 5.1 introduce a light-cone and interpret these relations in terms of space-like and time-like.

Introduce the notation

$$X = \begin{pmatrix} t \\ r \end{pmatrix}, \quad \Xi = \begin{pmatrix} dt \\ dr \end{pmatrix}, \quad Q = \begin{pmatrix} \cos(\pi\gamma) & i \sin(\pi\gamma) \\ i \sin(\pi\gamma) & \cos(\pi\gamma) \end{pmatrix} \quad q = e^{2\pi i \gamma}.$$

Then Q is unitary. The commutation relations in $\Omega^*(\mathcal{A})$ can be written in the form

$$X^t(Q\sigma_2)X = 0, \quad X\Xi^t = \Xi(Q^2X)^t, \quad \Xi^tQ\Xi = 0. \quad (2.16)$$

The σ_2 is the second Pauli matrix.

There are alternative $*$ -structures which require a real q . One can impose the conditions $u^* = v$, $v^* = u$. In terms of the original variables x and y this implies that

$$x^* = \pm \tilde{q}^{1/2}xy^{-1}, \quad y^* = y.$$

It follows that the frame satisfies

$$(\theta^1)^* = \theta^2, \quad (\theta^2)^* = \theta^1$$

and so one can introduce a real frame by taking the real and imaginary parts or consider the resulting structure as a q -deformed complex line. This is better with the change of generators

$$t = \frac{1}{\sqrt{2}}(u + v), \quad r = \frac{i}{\sqrt{2}}(u - v). \quad (2.17)$$

It is equivalent to a replacement $\gamma \mapsto i\gamma$ in the formula (2.16).

3 Representations

An extensive discussion of the $*$ -representations of the algebra \mathcal{A} for $|\tilde{q}| = 1$ and $q \neq \pm 1$ has been given ²⁸. We recall parts of it to illustrate our interpretation of the geometry. It is easy to see that there can be no normed basis with u or v diagonal. Suppose in fact that there is a basis with $v|j\rangle = v_j|j\rangle$. Since v is hermitian the eigenvalue $v_j \in \mathbb{R}$. Using the commutation relations one sees that $v(u|j\rangle) = q^{-1}v_j(u|j\rangle)$ and so $u|j\rangle$ is also an eigenvector with eigenvalue $q^{-1}v_j \notin \mathbb{R}$. One concludes therefore that $u|j\rangle \notin \mathcal{H}$. More specifically one can consider $\mathcal{H} = L^2(\mathbb{R})$ with the plane-wave basis $|k\rangle = e^{ikx}$. The operator $u = -i\partial_x$ is hermitian on a dense subspace of \mathcal{H} and diagonal: $u|k\rangle = k|k\rangle$. We can formally set

$$v|k\rangle = |qk\rangle = e^{-iqkx}$$

in order to have the correct commutation relations but u is not properly defined on the plane-wave basis.

As solution to this problem we restrict our representation space to the positive real line \mathbb{R}^+ with free boundary condition at $x = 0$. The Laplace transform replaces the Fourier transform and so we choose as basis $|k\rangle = e^{-kx}$ for $k \in \mathbb{C}$ with $\Re k > 0$. We need in fact represent only one (at a time) of the four regions defined by the light ‘cone’ and we choose the one defined by $\epsilon_1 = \epsilon_2 = 1$. Our sign conventions were partly dictated by the desire that this be the forward light-cone. We choose ²⁸ then two positive real numbers α and β with $\alpha\beta = \gamma$ and we define on the Hilbert space $L^2(\mathbb{R}^+)$

$$(uf)(x) = f(x + i\beta), \quad (vf)(x) = e^{-2\pi\alpha x}f(x).$$

Both u and v are formally hermitian and bounded. It is more convenient to express them in terms of the Laplace transform, which we recall is given by

$$F(k) = (Lf)(k) = \int_0^\infty f(x)e^{-kx}dx, \quad f(x) = (L^{-1}F)(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(k)e^{kx}dk$$

where a depends on the growth rate of the function. We have then

$$(uF)(k) \equiv (L(uf))(k) = e^{i\beta k}F(k), \quad (vF)(k) \equiv (L(vf))(k) = F(k + 2\pi\alpha).$$

In particular these transformation formulae are valid on the basis $|k\rangle = e^{-kx}$. The operators u and v are well-defined and positive for $\Re k > 0$.

4 The metrics and their connections

We now determine some possible metrics and metric compatible connections on the real quantum plane. We require them to fulfill all or at least part of the conditions listed in section 1, namely (1.18), (1.29), (1.25), (1.23), (1.27), (1.28).

To shorten the notation we shall often perform the following change of index notation: $(11, 12, 21, 22) \rightarrow (1, 2, 3, 4)$. Then the condition (1.23) can be written in the matrix form

$$\begin{pmatrix} S^1_1 & S^1_2 & S^1_3 & S^1_4 \\ S^2_1 & S^2_2 & S^2_3 & S^2_4 \\ S^3_1 & S^3_2 & S^3_3 & S^3_4 \\ S^4_1 & S^4_2 & S^4_3 & S^4_4 \end{pmatrix} \times (S_{(g)}) = \begin{pmatrix} g^1 & 0 & g^3 & 0 \\ 0 & g^1 & 0 & g^3 \\ g^2 & 0 & g^4 & 0 \\ 0 & g^2 & 0 & g^4 \end{pmatrix} \quad (4.1)$$

where we have introduced the matrix $S_{(g)}$ defined by

$$S_{(g)} = \begin{pmatrix} S^1_1 g^1 + S^1_2 g^3 & S^1_3 g^1 + S^1_4 g^3 & S^3_1 g^1 + S^3_2 g^3 & S^3_3 g^1 + S^3_4 g^3 \\ S^1_1 g^2 + S^1_2 g^4 & S^1_3 g^2 + S^1_4 g^4 & S^3_1 g^2 + S^3_2 g^4 & S^3_3 g^2 + S^3_4 g^4 \\ S^2_1 g^1 + S^2_2 g^3 & S^2_3 g^1 + S^2_4 g^3 & S^4_1 g^1 + S^4_2 g^3 & S^4_3 g^1 + S^4_4 g^3 \\ S^2_1 g^2 + S^2_2 g^4 & S^2_3 g^2 + S^2_4 g^4 & S^4_1 g^2 + S^4_2 g^4 & S^4_3 g^2 + S^4_4 g^4 \end{pmatrix}. \quad (4.2)$$

Using the expression (2.13) for P , the condition (1.25) becomes

$$g^2 = qg^3. \quad (4.3)$$

The consistency condition (1.16) is equivalent to the conditions

$$S^1_3 = qS^1_2, \quad S^2_3 = q(S^2_2 + 1), \quad S^3_3 = qS^3_2 - 1, \quad S^4_3 = qS^4_2. \quad (4.4)$$

The equations to be solved then are Equations (4.1), (4.3) and (4.4). We are especially interested in real solutions, which satisfy therefore also (1.27) and (1.28). We have found that there are several types of solutions, four of which we shall describe in the following subsections. One can show that there are no solutions with $\tau = 2$. A complete classification has been given ²⁹ of the solutions to the braid equation as well ^{30, 31} as of those which satisfy a weaker modified equation.

If one considers locality as of importance only in the commutative limit then there is no restriction on the coefficients of the metric, except that they be local functions in this limit. If one considers locality as of importance even before the limit but is willing to accept a metric which is real and symmetric only in the commutative limit then the most general line element one can write is of the form

$$ds^2 = g_{ij}\theta^i \otimes \theta^j.$$

The g_{ij} is a real symmetric matrix (in the sense we have defined it) and the moving frame θ^i is defined by

$$\theta^1 = vu^{-1}du, \quad \theta^2 = uv^{-1}dv.$$

The line element (1.30) becomes then

$$ds^2 = g_1v^2u^{-2}du^2 + 2g_2dudv + g_4u^2v^{-2}dv^2. \quad (4.5)$$

The product here is the symmetrized tensor product; not the exterior product.

The associated metric connection is given by the structure functions

$$C^1_{12} = u^{-1}, \quad C^2_{12} = -v^{-1}.$$

If we interpret the matrix g_{ij} as the components of the Killing metric on $SO(2)$ or $SO(1,1)$ then we can use it to calculate the connection form. The result will be of the form

$$\omega^i_j = A^i_{jk}u^{-1}\theta^k + B^i_{jk}v^{-1}\theta^k$$

with $g_{ik}\omega^k_j$ antisymmetric in the two indices. The Gaussian curvature K is a second-order homogeneous polynomial in the variables u^{-1} and v^{-1} :

$$K = \kappa_{11}u^{-2} + 2\kappa_{12}u^{-1}v^{-1} + \kappa_{22}v^{-2}.$$

4.1 Solution I

A 1-parameter family of solutions of conditions (1.18), (1.25), (1.23), can be found with a Minkowski-signature metric. For the particular value $\zeta = 0$ of the parameter also the braid relation (1.29) and the reality conditions (1.27), (1.28) are fulfilled. These are the most interesting solutions.

With the convenient normalization of the metric so that $g^3 = q^{-1/2}$ the flip is given by the matrix

$$S = \begin{pmatrix} q & -q^{-1/2}\zeta & -q^{1/2}\zeta & q^{-1}(q^2 - 1)^{-1}\zeta^2(q^2 + 1) \\ 0 & 0 & q & -q^{-1/2}\zeta \\ 0 & q^{-1} & 0 & q^{-3/2}\zeta \\ 0 & 0 & 0 & q^{-1} \end{pmatrix},$$

where $\zeta \in \mathbb{C}$. It tends to the ordinary flip as $q \rightarrow 1$ if $\zeta = 0$; only for $\zeta = 0$ it is a solution to the braid equation (1.29). The corresponding metric is given by

$$g^{ij} = \begin{pmatrix} (q - 1)^{-1}\zeta & q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix}. \quad (4.6)$$

From (4.3) one sees that it is σ -symmetric for all g^1 and real if $g^1 = 0$ (i.e. $\zeta = 0$). In this case S is given by

$$S = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \quad (4.7)$$

The σ and π are related as in (1.17) with $T^{ij} := \tau(\theta^i \otimes_{\mathcal{A}} \theta^j)$

$$T = \begin{pmatrix} 1+q & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1+q^{-1} \end{pmatrix}. \quad (4.8)$$

The fact that T is not proportional to the identity is due to the fact that the map $(1+\sigma)/2$ is not a projector and that we would like it to act as such and be the complementary to π . The metric matrix is of indefinite signature and in ‘light-cone’ coordinates. If we use the expression $q = e^{2\pi i \eta}$ we find that

$$g_S^{ij} = \cos(\pi\eta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_A^{ij} = i \sin(\pi\eta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.9)$$

The inverse metric components are defined by the equation

$$g_{ij} g^{jk} = \delta_i^k.$$

This matrix also can be split. If we rescale so that the symmetric part is of the standard form we find

$$(\eta_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (B_{ij}) = i \tan(\pi\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For the choice (4.7) of the flip (i.e. for $\zeta = 0$) the metric connection (1.20) is given by

$$(\omega^i_j) = (1-q) \begin{pmatrix} 1 & 0 \\ 0 & -q^{-1} \end{pmatrix} \theta,$$

and has vanishing curvature, because of the identities (2.11) and (1.29). This can be shown by an argument already used in ^{33, 34}. In other words, in this case the quantum plane is flat. In the commutative limit the line element is given by

$$ds^2 = g_{ij} \theta^i \otimes \theta^j = 2\theta^1 \otimes \theta^2 = 2du \otimes dv = dt^2 - dr^2.$$

The frame is singular along the light cone through the origin [see (2.9)]. Suppose $\epsilon_1 = \epsilon_2 = 1$. If in a representation one forces x and y to be hermitian then the u and v must be positive operators. One concludes then that $t > |r|$; the geometry describes only the forward light-cone through the origin. The other three regions are given by the other three possible combinations of signs.

4.2 Solution II

A family of solutions defined by flips which are solutions to (1.18), (1.27), but not to the braid equation (1.29) is given by

$$S = \begin{pmatrix} -q^2 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & -q^{-2} & -1 - q^{-1} & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \quad (4.10)$$

The metric is given again by (4.6) with $\zeta = 0$, and fulfills (1.23), (1.25), but not (1.28). The metric connection (1.20) is

$$(\omega^i_j) = (1 + q^2) \begin{pmatrix} 1 & 0 \\ 0 & q^{-2} \end{pmatrix} \theta + (1 + q^{-1}) \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \lambda_1 \theta^2 + (q + 1) \begin{pmatrix} q & 0 \\ 0 & q^{-2} \end{pmatrix} \lambda_2 \theta^2.$$

The curvature Curv is equal to

$$\Omega^i_j = -(q^2 - 1)q^{-3}(1 + q + q^2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (\lambda_1)^2 \theta^1 \theta^2.$$

It diverges as $(q - 1)^{-1}$ when $q \rightarrow 1$. This is then the case of a regular metric which has a singular metric connection.

4.3 Solution III

A third family,

$$S = \frac{1}{q^2 + 1} \begin{pmatrix} 2q & 0 & 0 & 1 - q^2 \\ 0 & 1 - q^2 & 2q & 0 \\ 0 & 2q & q^2 - 1 & 0 \\ q^2 - 1 & 0 & 0 & 2q \end{pmatrix}, \quad (4.11)$$

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

fulfills (1.18), (1.25), (1.23), the reality condition (1.27) but not the one (1.28) nor the braid relation (1.29). The latter are fulfilled for $q = \pm 1$. For $q = -1$ this means the connection form is imaginary in the usual sense of the word (since so are the λ_i).

The compatible connection (1.20) form is

$$(\omega^i_j) = \frac{(q - 1)^2}{q^2 + 1} \delta^i_j \theta + \frac{q^2 - 1}{q^2 + 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\lambda_2 \theta^1 + \lambda_1 \theta^2).$$

The curvature 2-form is

$$(\Omega^i_j) = \frac{(q^2 - 1)}{(q^2 + 1)^2} \left\{ -q^{-1}(q^2 - 1)^2 \delta^i_j \lambda_1 \lambda_2 + 2(q - 1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ((\lambda_1)^2 + (\lambda_2)^2) \right\} \theta^1 \theta^2.$$

In the limit $q \rightarrow 1$ this becomes

$$(\Omega^i_j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (u^{-2} + v^{-2}) \theta^1 \theta^2.$$

4.4 The \hat{R} -matrix ‘solution’

Finally one might ask whether one can find a solution (S, g) using the formalism of Faddeev et al.³², as has been done^{33, 34} for the q -euclidean ‘spaces \mathbb{R}_q^n ’ with $n > 2$. This would imply an S proportional to the braid matrix \hat{R} of $SL_q(2)$ or to its inverse. One can show that there is a solution only if one admits non-symmetry metrics.

We recall that the braid matrix which defines the Hopf algebra $SL_q(2)$

$$\hat{R}_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

fulfills the braid relation, admits the projector decomposition

$$\hat{R}_q = qP_{s,q} - q^{-1}P_{a,q}$$

and fulfills the (1.29) relations

$$\hat{R}_q^{\pm 1 ij}{}_{hk} \varepsilon_q^{kl} \hat{R}_q^{\pm 1 rs}{}_{jl} = q^{\mp 1} \varepsilon_q^{ir} \delta_h^s, \quad \hat{R}_q^{\pm 1 ij}{}_{hk} \varepsilon_q^{hk} = -q^{\mp 1} \varepsilon_q^{ij}, \quad (4.12)$$

where ε_q^{ij} is the q -deformed epsilon tensor

$$\varepsilon_q^{ij} = \begin{pmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{pmatrix}.$$

So one finds

$$P_{a,q}{}^{ij}{}_{hk} = (\varepsilon^{lm} \varepsilon_{lm})^{-1} (\varepsilon^{ij} \varepsilon_{hk}) = \frac{1}{q + q^{-1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By a straightforward computation one can check that (2.12) can be given the form (1.4) by setting

$$P = P_{a,q^{-1}}.$$

The first relation in (4.12) suggests that we make the Ansatz $S \propto \hat{R}_{q^{-1}}^{\pm 1}$, $g^{ij} \propto \varepsilon_{q^{-1}}^{ij}$, so that we can fulfill (1.23) at least up to a conformal factor. Equation (1.16) fixes the first proportionality constant to be either

$$S = q^{-1} \hat{R}_{q^{-1}} \quad \text{or} \quad S = q (\hat{R}_{q^{-1}})^{-1}$$

which respectively imply that

$$S^{im}{}_{ln} g^{np} S^{jk}{}_{mp} = q^{-1} g^{ij} \delta_l^k \quad S^{im}{}_{ln} g^{np} S^{jk}{}_{mp} = q g^{ij} \delta_l^k, \quad (4.13)$$

i.e. we indeed fulfill (1.23) only up to a conformal factor $q^{\pm 1}$, and

$$S^{ij}{}_{hk} g^{hk} = -g^{ij}. \quad (4.14)$$

This ‘antisymmetry’ relation is to be contrasted with Equation (1.24), which, with the above choice of S , amounts to replacing at the rhs of (4.14) -1 respectively by q^{-2} or q^2 , as can be seen writing P as a combination of S and of the identity matrix. Using the fact that $|q| = 1$ and $\hat{R}_{q^{-1}}^{ij}{}_{hk} = \hat{R}_q^{-1ji}{}_{kh}$ ³² one can easily see that the reality conditions (1.27) and (1.28) are satisfied. The curvature (1.21) can be easily calculated to be zero using the conditions $K^{ij} = 0$ and $F_{ij}^h = 0$ as well as the fact that P_q is a polynomial in S , which it turn fulfils the braid equation.

4.5 Other ‘solutions’

There are a certain number of partial solutions which are unsatisfactory for some reason or other. As an example, to underline the possibility of exotic metrics which are both symmetric and anti-symmetric according to our definitions, we consider σ defined by the matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & \zeta \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \zeta^{-1} & 0 & 0 & 0 \end{pmatrix}$$

where $\zeta \in \mathbb{R}$ is a parameter. This value of S is a solution to the braid equation. The σ and π are related as in (1.17) with (using the same conventions)

$$\mathbf{1} + S = T = \begin{pmatrix} 1 & 0 & 0 & \zeta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \zeta^{-1} & 0 & 0 & 1 \end{pmatrix}. \quad (4.15)$$

This means that τ is not invertible and the case is degenerate. The unpleasant thing here is that $(1 + \sigma)/2$ and π do not add up to the identity map. The metric is given by

$$g^{ij} = i \begin{pmatrix} 1 & 0 \\ 0 & -\zeta^{-1} \end{pmatrix}. \quad (4.16)$$

One has $\tau = 1 + \sigma$ and the flip is degenerate. Instead of interchanging g^2 and g^3 as does the ordinary flip, it interchanges g^1 and g^4 . It also changes the sign, which accounts for the i in the metric components. Also $g \circ (1 + \sigma) = 0$ so in a certain sense the metric has vanishing symmetric as well as antisymmetric parts. We refer to σ nonetheless as a ‘flip’ because it satisfies (1.16).

The linear connection (1.19) is given by

$$\omega_j^i = \delta_j^i \theta + \begin{pmatrix} 0 & 1 \\ -\zeta^{-1} & 0 \end{pmatrix} (\zeta \lambda_1 \theta^2 - \lambda_2 \theta^1)$$

The curvature is given by

$$\Omega^i_j = q^{-1}(q^2 - 1)\delta^i_j \lambda_1 \lambda_2 \theta^1 \theta^2$$

The connection is singular in the commutative limit as is the curvature. Because of (1.24) it cannot be satisfied for any curvature which is proportional to the metric.

5 Jordanian deformation

It has been shown recently (See, for example, Aneva *et al.* ³¹) that the jordanian deformation is a singular limit of a family of q deformations. The transformation from the set of generators of one algebra to the other has also been studied in some detail ³⁵. We can now discuss to what extent the limit can be understood in a geometric manner. We recall that the jordanian deformation is defined using a parameter h and that the generators (x', y') satisfy the commutation relations $[x', y'] = hy'^2$. The differential calculus is given by two elements λ'_a similar to the λ_a which satisfy the $SL(2, \mathbb{R})$ relation $[\lambda'_1, \lambda'_2] = \lambda'_1$, a relation which is not quadratic. This must be compared with the quadratic relation $\lambda_1 \lambda_2 = q^{-1} \lambda_2 \lambda_1$ satisfied by the elements (2.10). We must find a smooth map from one algebra into the other, that is, one which respects the commutation relations between the elements which define the derivations dual to the frame. Consider ³⁵ the map

$$\lambda'_1 = h_0^{-1} \lambda_1, \quad \lambda'_2 = h_0^{-1} \lambda_2 - \frac{1}{2} h^{-1} h_0, \quad h_0 = \frac{2h}{1-q}. \quad (5.1)$$

This change defines a deformation of the differential calculus. From the commutation relations of the λ_i we deduce that

$$[\lambda'_1, \lambda'_2] = h_0^{-2} [\lambda_1, \lambda_2] = h_0^{-2} (1-q) \lambda_1 \lambda_2 = \lambda'_1 + (1-q) \lambda'_1 \lambda'_2.$$

In the (singular) limit when $q \rightarrow 1$ the differential calculus tends to that of the jordanian deformation.

The relations between the two calculi can be written in terms of a diagram

$$\begin{array}{ccc} (x, y) & \longrightarrow & (u, v) = (\epsilon_2 q^{1/2} x^2, \epsilon_1 x^2 y^{-2}). \\ \downarrow & & \downarrow \\ (x', y') & \longrightarrow & (u', v') = (x' y'^{-1} + \frac{1}{2} h, y'^{-2}) \end{array} \quad (5.2)$$

The two horizontal arrows are changes of generators. The two vertical ones define a map between the two deformations. In terms of the generators u and v and their analogues ³⁶ u' and v' for the jordanian deformation, the map (5.1) can be written as

$$u' = qu^{-1} - h_0, \quad v' = -qv^{-1}$$

with $h_0 \rightarrow \infty$. It has been shown ³⁶ that the local metric on the jordanian deformation is that of Lobachevsky. This must be a limit of one of the family of metrics (4.5). The

Lobachevsky metric can be described with the line element $ds'^2 = v'^{-2}(du'^2 + dv'^2)$. To compare we write (4.5) in the primed variables:

$$ds^2 = (u' + h_0)^{-2} v'^{-2} [q^2 g_1 du'^2 - 2g_2 du' dv' + q^{-2} g_4 dv'^2].$$

We see than that we must choose $g_2 = 0$ and let $g_1, g_4 \rightarrow \infty$ with the constraint

$$g_1 h_0^{-2} = g_4 h_0^{-2} = 1.$$

The quantum-plane metric belongs to the family III. Another interesting metric obtained in the same limit is with $g_1 = g_4 = 0$ and $g_2 \rightarrow \infty$ so that $g_2 h_0^{-2} = 1$:

$$ds^2 = -2v'^{-2} du' dv' = -2du' dv.$$

This solution belongs to the family I.

6 Patching

Let us consider now the solutions I found in section 4.1. To each of the four regions defined by the light cone through the origin in two dimensions we have associated an algebra, a differential calculus and a metric, but none is complete as ‘manifold’. From the form of the metric we see that this can be done using the generators (t, r) or (u, v) but that the generators (x, y) are singular on the cone.

The patching is done²⁸ by extending the domain of definition of u for example to negative eigenvalues. The frame θ^i is also singular on the cone but the equivalent frame du^i is quite regular. We can write $\theta^i = \Lambda^i_j du^j$ where

$$\Lambda^i_j = \sqrt{q} \begin{pmatrix} vu^{-1} & 0 \\ 0 & uv^{-1} \end{pmatrix}$$

is a local Lorentz transformation in the commutative limit.

7 Discussion

We have given a partial classification of the solutions to the three conditions of metric compatibility (1.23), symmetry (1.24) and the consistency condition (1.16), as well as the reality conditions (1.27), (1.28), and the braid relation (1.29), without due regard to quantum covariance. In fact we could show that there was no solution which respected a coaction of the quantum group. A similar problem was found by Cotta-Ramusino & Rinaldi in trying to construct holonomy groups³⁷. Written in terms of the components in the frame basis one sees that S^{ij}_{kl} has 16 unknowns and g^{ij} has 4 unknowns. The condition (1.16) gives 4 equations and metric compatibility gives 16 equations. So a naive computation would say that the solution is unique up to a rescaling of g^{ij} , which is not fixed by the equation. We have indeed found a finite set of solutions.

Another conclusion concerns the uniqueness of the vacuum. It has been claimed ³⁸ that within the context of the present formalism there is essentially a unique differential calculus which has associated to it a given metric, unique that is up to a choice of norm on the frame. This statement needs qualification since we have here shown that the quantum plane is naturally endowed with the Lorentz-signature flat metric and it is known that the same is true of the Heisenberg algebra with its natural differential calculus.

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